

Clifford Algebra of Spacetime and the Conformal Group

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Received March 25, 2003

We demonstrate the emergence of the conformal group $SO(4,2)$ from the Clifford algebra of spacetime. The latter algebra is a manifold, called Clifford space, which is assumed to be the arena in which physics takes place. A Clifford space does not contain only points (events), but also lines, surfaces, volumes, etc., and thus provides a framework for description of extended objects. A subspace of the Clifford space is the space whose metric is invariant with respect to the conformal group $SO(4,2)$ which can be given either passive or active interpretation. As advocated long ago by one of us, active conformal transformations, including dilatations, imply that sizes of physical objects can change as a result of free motion, without the presence of forces. This theory is conceptually and technically very different from Weyl's theory and provides when extended to a curved conformal space a resolution of the long standing problem of realistic masses in Kaluza–Klein theories.

KEY WORDS: Clifford algebra; conformal group; geometric calculus.

1. INTRODUCTION

Extended objects such as membranes or branes of any dimension are nowadays subject of extensive studies. A deeper geometric principle behind such theories remains to be fully explored. It has been found that various string theories are different faces of a conjectured single theory, called M-theory which has not yet been rigorously formulated. In a number of recent works branes have been considered from the point of view of the geometric calculus based on Clifford algebra (Castro, 1998, 2000a,b, 2001; Pavšič, 2001a). The latter is a very useful tool for description of geometry and physics (Ablamowitz *et al.*, 2000; Castro, 1998, 2000a,b, 2001; Castro and Pavšič, 2002; Hestenes, 1966, 1984; Lasenby *et al.*, 1993a,b; Lounesto, 2001; Pavšič, 2001a,b; Pezzaglia, 1993, 1996, 1997, 1999; Pezzaglia and Adams, 1997; Pezzaglia and Differ, 1993). A space (or spacetime)

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consists of points (or events). But besides points there are also lines, surfaces, volumes, etc. Description of such geometric objects turns out to be very elegant if one employs multivectors which are the outer products of vectors. All those objects are elements of Clifford algebra. Since in physics we do not consider point particles only, but also extended objects, it appears natural to consider Clifford algebra as an arena in which physics takes place. Clifford algebra as an arena for physics has been called *pandimensional continuum* (Pezzagila, 1993, 1996, 1997, 1999; Pezzagial and Adams, 1997; Pezzagila and Differ, 1993) or *C-space* (Castro, 2000a,b, 2001).

In this paper we report about a possibly far reaching observation that Clifford algebra of 4-dimensional spacetime contains conformal group $SO(4,2)$ as a special case. It was proposed long time ago (Pavšič, 1977, 1980) that conformal space can serve as the arena for physics which involves active dilatations and dilatational motions, and that a curved conformal space is a possible realization for Kaluza–Klein theory. A remarkable property of such an approach to Kaluza–Klein theories is that the notorious problem of Planck mass does not occur, since the 4-dimensional mass is given by the expression $m = \sqrt{m_{00} + \pi_5 \pi_6}$ in which m_{00} is the invariant mass in six dimensions, π_6 the electric charge and π_5 the dilatational momentum (taken to be zero for the ordinary electron). However in that old work it was not yet realized that the variables κ and λ entering the description of $SO(4,2)$ —which obviously were not just the ordinary extra dimensions—are natural ingredients of the Clifford algebra of 4-dimensional spacetime: they are coordinates of *C-space*.

The idea to allow objects to change in size when they move in spacetime is in fact very old and dates back to Weyl [1918]. However, our proposal, initiated in refs. (Pavšič, 1977, 1980), differs from that of Weyl both conceptually and technically, and is free from Einstein's well-known criticism. In Weyl's geometry one is gauging the local scale transformations by introducing a gauge field, identified with the electromagnetic field potential, such that sizes of objects (and the rate at which clocks tick) are path dependent. Einstein pointed out that this would effect spectral lines emitted by the atoms which had followed different paths in spacetime and then brought together. A result would be blurred spectra with no distinctive spectral lines, contrary to what we observe. In our approach we are not gauging the local scale transformations and scale changes of objects are not due to the different paths they traverse. Scale is postulated as an extra degree of freedom, analogous to position. If the observer chooses to be in the same scale-frame of reference as the particular object he observes (for more detailed description see refs. (Pavšič, 1977, 1980)), no scale changes for that object are observed (with respect to the observer) in its ordinary spacetime motion. However, in general, our theory predicts that in a given scale-frame of reference chosen by the observer, different objects can have different scales (i.e., different sizes) and consequently emit spectral lines whose wavelengths are shifted by the corresponding scale factors. When considering the quantized theory, not only position but also scale has to be quantized (Pavšič, 1977, 1980) which affects not only relative positions but also relative scales of

bound objects, e.g., the atoms within a crystal. A crystal as a free object can have arbitrary position and scale, while the atoms bound within the crystal have the relative positions and scales as determined by the solutions of the Schrödinger equation (generalized to C-space). Possible astrophysical implications of scale as a degree of freedom were discussed (Pavšič, 1977, 1980).

2. ON GEOMETRY, CLIFFORD ALGEBRA AND PHYSICS

Since the appearance of the seminal books by Hestenes (1966, 1984) we have a very useful language and tool for geometry and physics, which is being recognized by an increasing number of researchers (Ablamowitz *et al.*, 2000; Castro and Pavšič, 2002; Lasenby *et al.*, 1993a,b; Lounesto, 2001; Pavšič, 2001b; Pezzaglia, 1993, 1996, 1997, 1999; Pezzaglia and Adams, 1997; Pezzaglia and Differ, 1993). Although Clifford algebra is widely used and explored both in mathematics and physics, its full power for formulation of new physical theories has been recognized only relatively recently (Castro, 2000a,b, 2001; Pavšič, 2001a,b; Pezzaglia, 1993, 1996, 1997, 1999; Pezzaglia and Adams, 1997; Pezzaglia and Differ, 1993). For the reasons of self-consistency we will provide a brief introduction into the geometric calculus based on Clifford algebra and point out how Minkowski spacetime can be generalized to Clifford space.

The starting observation is that the basis vectors e_μ in an n -dimensional space V_n satisfy the Clifford algebra relations

$$e_\mu \cdot e_\nu \equiv \frac{1}{2}(e_\mu e_\nu + e_\nu e_\mu) = g_{\mu\nu} \quad (1)$$

where $g_{\mu\nu}$ is the metric of V_n . The dot denotes the *inner product* of two vectors. It is the symmetric part of the noncommutative *Clifford* or *geometric product*

$$e_\mu e_\nu = e_\mu \cdot e_\nu + e_\mu \wedge e_\nu \quad (2)$$

whose antisymmetric part is the *wedge* or *outer product*

$$e_\mu \wedge e_\nu \equiv \frac{1}{2}(e_\mu e_\nu - e_\nu e_\mu) \equiv \frac{1}{2!}[e_\mu, e_\nu] \quad (3)$$

While the inner product is a scalar, the outer product is a bivector. It denotes an *oriented area* of a 2-surface enclosed by a 1-loop. The precise shape of the loop is not determined. In a similar manner we can form higher *multivectors* or *r-vectors*

$$e_{\mu_1} \wedge e_{\mu_2} \wedge \cdots \wedge e_{\mu_r} \equiv \frac{1}{r!}[e_{\mu_1}, e_{\mu_2}, \dots, e_{\mu_r}] \quad (4)$$

by antisymmetrizing the Clifford product of r vectors. Such an object is interpreted geometrically to denote an oriented r -area of an r -surface enclosed by an $(r - 1)$ -loop. Multivectors are elements of the *Clifford algebra* C_n of V_n . An element of C_n is called a *Clifford number*.

In the geometric (Clifford) product (2) a scalar and a bivector occur in the sum. In general, a Clifford number is a superposition, called a *Clifford aggregate* or *polyvector*:

$$A = a \frac{1}{2!} a^\mu e_\mu + \frac{1}{3!} a^{\mu\nu} e_\mu \wedge e_\nu + \dots + \frac{1}{n!} a^{\mu_1 \dots \mu_n} e_{\mu_1} \wedge \dots \wedge e_{\mu_n} \quad (5)$$

In an n -dimensional space an n -vector is a multivector of the highest possible degree; an $(n + 1)$ -vector is identically zero.

Considering now a flat 4-dimensional spacetime with basis vectors γ_μ satisfying

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} \quad (6)$$

where $\eta_{\mu\nu}$ is the diagonal metric with signature $(+ - - -)$ Eq. (5) reads

$$D = d + d^\mu \gamma_\mu + \frac{1}{2!} \gamma_\mu \wedge \gamma_\nu + \frac{1}{3!} d^{\mu\nu\rho} \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho + \frac{1}{4!} d^{\mu\nu\rho\sigma} \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \wedge \gamma_\sigma \quad (7)$$

where $d, d^\mu, d^{\mu\nu}, \dots$, are scalar coefficients. The Clifford algebra in Minkowski space V_4 is called the *Dirac algebra*.

Let us introduce the symbol I for the unit element of 4-volume (pseudoscalar)

$$I \equiv \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = \gamma_0 \gamma_1 \gamma_2 \gamma_3, \quad I^2 = -1 \quad (8)$$

Using the relations

$$\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \wedge \gamma_\sigma = I \epsilon_{\mu\nu\rho\sigma} \quad (9)$$

$$\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho = I \epsilon_{\mu\nu\rho\sigma} \gamma^\sigma \quad (10)$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor, and introducing the new coefficients

$$S \equiv d, \quad V^\mu \equiv d^\mu, \quad T^{\mu\nu} \equiv \frac{1}{2} d^{\mu\nu} \quad (11)$$

$$C_\sigma \equiv \frac{1}{3!} d^{\mu\nu\rho} \epsilon_{\mu\nu\rho\sigma}, \quad P \equiv \frac{1}{4!} d^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} \quad (12)$$

we can rewrite D of Eq. (7) as the sum of a scalar, vector, bivector, pseudovector, and pseudoscalar:

$$D = S + V^\mu \gamma_\mu + T^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + C^\mu I_{\gamma_\mu} + P I \quad (13)$$

So far physics in spacetime has been predominantly using only the vector part of D . The full Clifford algebra or Dirac algebra have been used in relativistic quantum theory, but not in the classical special or general relativity, neither in the theory of strings and branes.

Assuming that fundamental physical objects are not point particles but extended objects such as strings and branes of arbitrary dimension, it has been proposed (Castro, 2000a,b, 2001; Pavšič, 2001a,b; Pezzaglia, 1993, 1996, 1997, 1999; Pezzaglia and Adams, 1997; Pezzaglia and Differ, 1993) that *physical quantities such as positions and velocities of those objects are polyvectors*. It was proposed to rewrite the known fundamental string and brane actions by employing polyvector coordinates

$$X = \frac{1}{r!} \sum_{r=0}^n X^{\mu_1 \dots \mu_r} \gamma_{\mu_1} \wedge \dots \wedge \gamma_{\mu_r} \equiv X^A E_A \tag{14}$$

Here we use a compact notation in which $X^A \equiv X^{\mu_1 \dots \mu_r}$ are *real* coordinates, and $E_A \equiv \gamma_{\mu_1} \wedge \dots \wedge \gamma_{\mu_r}$ basis vectors of the 2^n -dimensional *Clifford algebra* of spacetime. The latter algebra of spacetime positions and corresponding higher grade objects, namely oriented r -areas, is a manifold which is more general than spacetime. In the literature such a manifold has been named *pan-dimensional continuum* (Pezzaglia, 1993, 1996, 1997, 1999; Pezzaglia and Adams, 1997, Pezzaglia and Differ, 1993) or *Clifford space* or *C-space* (Castro, 2000a,b, 2001).

The infinitesimal element of position polyvector (14) is

$$dX = \frac{1}{r!} \sum_{r=0}^n dX^{\mu_1 \dots \mu_r} \gamma_{\mu_1} \wedge \dots \wedge \gamma_{\mu_r} \equiv dX^A E_A \tag{15}$$

We will now calculate its norm squared. Using the definition for the *scalar product* of two polyvectors A and B

$$A * B = \langle AB \rangle_0 \tag{16}$$

where $\langle \rangle_0$ means the scalar part of the geometric product AB , we obtain

$$|dX|^2 \equiv dX^\dagger * dX = dX^A dX^B G_{AB} = dX^A dX_A \tag{17}$$

Here

$$G_{AB} = E_A^\dagger * E_B \tag{18}$$

is the C -space metric and A^\dagger the reverse⁴ of a polyvector A .

For example, if the indices assume the values $A = \mu, B = \nu$, we have

$$G_{\mu\nu} = \langle e_\mu e_\nu \rangle_0 = e_\mu \cdot e_\nu = g_{\mu\nu} \tag{19}$$

If $A = [\mu\nu], B = [\alpha\beta]$

$$\begin{aligned} G_{[\mu\nu][\alpha\beta]} &= \langle (e_\mu \wedge e_\nu)^\dagger (e_\alpha \wedge e_\beta) \rangle_0 = \langle (e_\mu \wedge e_\nu)^\dagger \cdot (e_\alpha \wedge e_\beta) \rangle_0 \\ &= (e_\mu \cdot e_\alpha)(e_\nu \cdot e_\beta) - (e_\nu \cdot e_\alpha)(e_\mu \cdot e_\beta) = g_{\mu\alpha} g_{\nu\beta} - g_{\nu\alpha} g_{\mu\beta} \end{aligned} \tag{20}$$

⁴Reversion or, alternatively, *hermitian conjugation*, is the operation which reverses the order of all products of vectors in a decomposition of a polyvector A .

If $A = \mu, B = [\alpha\beta]$

$$G_{\mu[\alpha\beta]} = \langle e_\mu(e_\alpha \wedge e_\beta) \rangle_0 = 0 \tag{21}$$

Explicitly we have

$$\begin{aligned} |dX|^2 &= \frac{1}{r!} \sum_{r=0}^n dX^{\mu_1 \dots \mu_r} dX_{\mu_1 \dots \mu_r} \\ &= ds^2 + dX^\mu dX_\mu + \frac{1}{2!} dX^{\mu_1 \mu_2} dX_{\mu_1 \mu_2} + \dots + \frac{1}{n!} dX^{\mu_1 \dots \mu_n} dX_{\mu_1 \dots \mu_n} \end{aligned} \tag{22}$$

If γ_μ are taken to be dimensionless so that $X^\mu, X^{\mu\nu}$, etc., have respectively dimensions of *length, length*², etc., then a suitable power of a length parameter has to be included in every term of Eq. (22). One natural choice is to take the length parameter equal to the Planck scale L_P . For simplicity reasons we may then use the system of units in which $L_P = 1$.

In 4-dimensional spacetime the vector (15) and its square (22) can be written as

$$dX = ds + dx^\mu \gamma_\mu \frac{1}{2} dx^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + d\tilde{x}^\mu I_{\gamma_\mu} + d\tilde{s} I \tag{23}$$

$$|dX|^2 = ds^2 + dx^\mu dx_\mu + \frac{1}{2} dx^{\mu\nu} dx_{\mu\nu} - d\tilde{x}^\mu d\tilde{x}_\mu - d\tilde{s}^2 \tag{24}$$

where we now use the lower case symbols for coordinates. The minus sign in the last two terms of the above quadratic form occurs because in 4-dimensional spacetime with signature $(+ - - -)$ we have $I^2 = (\gamma_0 \gamma_1 \gamma_2 \gamma_3)(\gamma_0 \gamma_1 \gamma_2 \gamma_3) = -1$, and $I^\dagger I = (\gamma_3 \gamma_2 \gamma_1 \gamma_0)(\gamma_0 \gamma_1 \gamma_2 \gamma_3) = -1$.

In Eq. (24) the line element $dx^\mu dx_\mu$ of the ordinary special or general relativity is replaced by the line element in Clifford space. A “square root” of such a generalized line element is dX of Eq. (23). The latter object is a *polyvector*, a differential of the coordinate polyvector field

$$X = s + x^\mu \gamma_\mu + \frac{1}{2} x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + \tilde{x}^\mu I_{\gamma_\mu} + \tilde{s} I \tag{25}$$

whose square is

$$|X|^2 = s^2 + x^\mu x_\mu + \frac{1}{2} x^{\mu\nu} x_{\mu\nu} - \tilde{x}^\mu \tilde{x}_\mu - \tilde{s}^2 \tag{26}$$

The polyvector X contains not only the vector part $x^\mu \gamma_\mu$, but also a *scalar part* s , *tensor part* $x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu$, *pseudovector part* $\tilde{x}^\mu I_{\gamma_\mu}$, and pseudoscalar part $\tilde{s} I$. Similarly for the differential dX .

When calculating the quadratic forms $|X|^2$ and $|dX|^2$ one obtains in 4-dimensional spacetime with pseudoeuclidean signature $(+ - - -)$ the minus sign

in front of the squares of the pseudovector and pseudoscalar terms. This is so, because in such a case the pseudoscalar unit square in flat spacetime is $I^2 = I^\dagger I = -1$. In four dimensions $I^\dagger = I$ regardless of the signature.

Instead of Lorentz transformations—pseudorotations in spacetime—which preserve $x^\mu x_\mu$ and $dx^\mu dx_\mu$ we have now more general rotations—rotations in C -space—which preserve $|X|^2$ and $|dX|^2$.

3. C-SPACE AND CONFORMAL TRANSFORMATIONS

From Eqs. (24) and (26) we see that a subgroup of the Clifford Group, or rotations in C -space is the group $SO(4,2)$. The transformations of the latter group rotate x^μ , s , \tilde{s} , but leave $x^{\mu\nu}$ and \tilde{x}^μ unchanged. Although according to our assumption physics takes place in full C -space, it is very instructive first to consider a subspace of C -space, that we shall call *conformal space* whose isometry group is $SO(4,2)$.

Coordinates can be given arbitrary symbols. Let us now use the symbol η^μ instead of x^μ , and η^5, η^6 instead of \tilde{s}, s . In other words, instead of (x^μ, \tilde{s}, s) we write $(\eta^\mu, \eta^5, \eta^6) \equiv \eta^a, \mu = 0, 1, 2, 3, a = 0, 1, 2, 3, 5, 6$. The quadratic form reads

$$\eta^a \eta_a = g_{ab} \eta^a \eta^b \tag{27}$$

with

$$g_{ab} = \text{diag}(1, -1, -1, -1, -1, 1) \tag{28}$$

being the diagonal metric of the flat 6-dimensional space, a subspace of C -space, parameterized by coordinates η^a . The transformations which preserve the quadratic form (27) belong to the group $SO(4,2)$. It is well known (Barut, and Haugen, 1972; Kastrup, 1962) that the latter group, when taken on the cone

$$\eta^a \eta_a = 0 \tag{29}$$

is identical to the 15-parameter group of conformal transformations in 4-dimensional spacetime (Bateman, 1910; Cunningham, 1909; Fulton *et al.*, 1962; Grillo, 1973; Mack, 1968; Niederle and Tolar, 1973; Wess, 1960).

Let us consider first the rotations of η^5 and η^6 which leave coordinates η^μ unchanged. The transformations that leave $-(\eta^5)^2 + (\eta^6)^2$ invariant are

$$\begin{aligned} \eta'^5 &= \eta^5 \text{ch } \alpha + \eta^6 \text{sh } \alpha \\ \eta'^6 &= \eta^5 \text{sh } \alpha + \eta^6 \text{ch } \alpha \end{aligned} \tag{30}$$

where α is a parameter of such pseudorotations.

Instead of the coordinates η^5, η^6 we can introduce (Barut and Haugen, 1972; Kastrup, 1962) new coordinates κ, λ according to

$$\kappa = \eta^5 - \eta^6 \tag{31}$$

$$\lambda = \eta^5 + \eta^6 \tag{32}$$

In the new coordinates the quadratic form (27) reads

$$\eta^a \eta_a = \eta^\mu \eta_\mu - (\eta^5)^2 - (\eta^6)^2 = \eta^\mu \eta_\mu - \kappa \lambda \quad (33)$$

The transformation (30) becomes

$$\kappa' = \rho^{-1} \kappa \quad (34)$$

$$\lambda' = \rho \lambda \quad (35)$$

where $\rho = e^\alpha$. This is just a dilation of κ and the inverse dilation of λ .

Let us now introduce new coordinates x^μ according x^μ to⁵

$$\eta^\mu = \kappa x^\mu \quad (36)$$

Under the transformation (36) we have

$$\eta'^\mu = \eta^\mu \quad (37)$$

but

$$x'^\mu = \rho x^\mu \quad (38)$$

The latter transformation is *dilatation* of coordinates x^μ .

Considering now a line element

$$d\eta^a d\eta_a = d\eta^\mu d\eta_\mu - d\kappa d\lambda \quad (39)$$

we find that *on the cone* $\eta^a \eta_a = 0$ it is

$$d\eta^a d\eta_a = \kappa^2 dx^\mu dx_\mu \quad (40)$$

even if κ is not constant. Under the transformation (34) we have

$$d\eta'^a d\eta'_a = d\eta^a d\eta_a \quad (41)$$

$$dx'^\mu dx'_\mu = \rho^2 dx^\mu dx_\mu \quad (42)$$

The last relation is a *dilatation* of the 4-dimensional line element related to coordinates x^μ . In a similar way also other transformations of the group $SO(4,2)$ that preserve (29) and (41) we can rewrite in terms of the coordinates x^μ . So we obtain—besides dilations—translations, Lorentz transformations, and special conformal transformations; altogether they are called *conformal transformations*. This is a well known old observation (Barut and Haugen, 1972; Kastrop, 1962) and we shall not discuss it further. What we wanted to point out here is that conformal group $SO(4,2)$ is a subgroup of the Clifford group.

⁵ These new coordinates x^μ should not be confused with coordinate x^μ used in Section 2.

4. ON THE PHYSICAL INTERPRETATION OF THE CONFORMAL GROUP $SO(4,2)$

In order to understand the physical meaning of the transformations (36) from the coordinates η^μ to the coordinates x^μ let us consider the following transformation in 6-dimensional space V_6 :

$$\begin{aligned}x^\mu &= \kappa^{-1} \eta^\mu \\ \alpha &= -\kappa^{-1} \\ \Lambda &= \lambda - \kappa^{-1} \eta^\mu \eta_\mu\end{aligned}\tag{43}$$

This is a transformation from the coordinates $\eta^a = (\eta^\mu, \kappa, \lambda)$ to the new coordinates $x^a = (x^\mu, \alpha, \Lambda)$. No extra condition on coordinates, such as (29), is assumed now. If we calculate the line element in the coordinates η^a and x^a , respectively, we find the following relation (Pavšič, 1980)

$$d\eta^\mu d\eta^\nu g_{\mu\nu} - d\kappa d\lambda = \alpha^{-2}(dx^\mu dx^\nu g_{\mu\nu} - d\alpha d\Lambda)\tag{44}$$

We can interpret a transformation of coordinates passively or actively. Geometric calculus clarifies significantly the meaning of passive and active transformations. Under a *passive transformation* a vector remains the same, but its components and basis vector change. For a vector $d\eta = d\eta^a \gamma_a$ we have

$$d\eta' = d\eta'^a \gamma'_a = d\eta^a \gamma_a = d\eta\tag{45}$$

with

$$d\eta'^a = \frac{\partial \eta'^a}{\partial \eta^b} d\eta^b\tag{46}$$

and

$$\gamma'_a = \frac{\partial \eta^b}{\partial \eta'^a} \gamma_b\tag{47}$$

Since the vector is invariant, so it is its square:

$$d\eta'^2 = d\eta'^a \gamma'_a d\eta'^b \gamma'_b = d\eta'^a d\eta'^b g'_{ab} = d\eta^a d\eta^b g_{ab}\tag{48}$$

From Eq. (47) we read that the well known relation between new and old coordinates:

$$g'_{ab} = \frac{\partial \eta^c}{\partial \eta'^a} \frac{\partial \eta^d}{\partial \eta'^b} g_{cd}\tag{49}$$

Under an *active transformation* a vector changes. This means that in a fixed basis the components of a vector change:

$$d\eta' = d\eta'^a \gamma_a\tag{50}$$

with

$$d\eta'^a = \frac{\partial \eta'^a}{\partial \eta^b} d\eta^b \quad (51)$$

The transformed vector $d\eta'$ is different from the original vector $d\eta = d\eta^a \gamma_a$. For the square we find

$$d\eta'^2 = d\eta'^a d\eta'^b g_{ab} = \frac{\partial \eta'^a}{\partial \eta^c} \frac{\partial \eta'^b}{\partial \eta^d} d\eta^c d\eta^d g_{ab} \quad (52)$$

i.e., the transformed line element $d\eta'^2$ is different from the original line element.

Returning now to the coordinate transformation (43) with the identification $\eta'^a = x^a$, we can interpret Eq. (44) passively or actively.

In the *passive interpretation* the metric tensor and the components $d\eta^a$ change under a transformation, so that in our particular case the relation (48) becomes

$$dx^a dx^b g'_{ab} = \alpha^{-2} (dx^\mu dx^\nu g_{\mu\nu} - d\alpha d\Lambda) = d\eta^a d\eta^b g_{ab} = d\eta^\mu d\eta^\nu g_{\mu\nu} - d\kappa d\lambda \quad (53)$$

with

$$g'_{ab} = \alpha^{-2} \begin{pmatrix} g_{\mu\nu} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}, \quad g_{ab} = \begin{pmatrix} g_{\mu\nu} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} \quad (54)$$

In the above equation the same infinitesimal distance squared is expressed in two different coordinates η^a or x^a .

In *active interpretation*, only $d\eta^a$ change, while the metric remains the same, so that the transformed element is

$$\begin{aligned} dx^a dx^b g_{ab} &= dx^\mu dx^\nu g_{\mu\nu} - d\alpha d\Lambda = \kappa^{-2} d\eta^a d\eta^b g_{ab} \\ &= \kappa^{-2} (d\eta^\mu d\eta^\nu g_{\mu\nu} - d\kappa d\lambda) \end{aligned} \quad (55)$$

The transformed line element $dx^a dx_a$ is physically different from the original line element $d\eta^a d\eta_a$ by a factor $\alpha^2 = \kappa^{-2}$.

A rotation (30) in the plane (η^5, η^6) (i.e., the transformation (34),(35) of (κ, λ)) manifests in the new coordinates x^a as a *dilatation* of the line element $dx^a dx_a = \kappa^{-2} d\eta^a \eta_a$:

$$dx'^a dx'_a = \rho^2 dx^a dx_a \quad (56)$$

All this is true in the full space V_6 . On the cone $\eta^a \eta_a = 0$ we have $\Lambda = \lambda - \kappa \eta^\mu \eta_\mu = 0$, $d\Lambda = 0$ so that $dx^a dx_a = dx^\mu dx_\mu$ and we reproduce the relations (42) which is a dilatation of the 4-dimensional line element. It can be interpreted either passively or actively. In general, the pseudorotations in V_n , i.e., the transformations of the 15-parameter group $SO(4,2)$ when expressed in terms

of coordinates x^a , assume on the cone $\eta^a \eta_a = 0$ the form of the ordinary conformal transformations. They all can be given the active interpretation Pavšič (1977, 1980).

5. CONCLUSION

We started from the new paradigm that physical phenomena actually occur not in spacetime, but in a larger space, the so-called *Clifford space* or *C-space* which is a manifold associated with the Clifford algebra generated by the basis vectors γ_μ of spacetime. An arbitrary element of Clifford algebra can be expanded in terms of the objects E_A , $A = 1, 2, \dots, 2^D$, which include, when $D = 4$, the scalar unit 1, vectors γ_μ , bivectors $\gamma_\mu \wedge \gamma_\nu$, pseudovectors $I_{\gamma\mu}$, and the pseudoscalar unit $I \equiv \lambda_5$. *C-space* contains 6-dimensional subspace V_6 spanned⁶ by $\mathbf{1}$, γ_μ , and γ_5 . The metric of V_6 has the signature $(+ - - - - +)$. It is well known that the rotations in V_6 , when taken on the conformal cone $\eta^a \eta_a = 0$, are isomorphic to the nonlinear transformations of the conformal group in spacetime. Thus we have found out that *C-space* contains—as a subspace—the 6-dimensional space V_6 in which the conformal group acts linearly. From the physical point of view this is an important and, as far as we know, a novel finding, although it might look mathematically trivial. So far it has not been clear what could be a physical interpretation of the 6-dimensional conformal space. Now we see that it is just a subspace of Clifford space.

We take *C-space* seriously as an arena in which physics takes place. The theory is a very natural, although not trivial, extension of the special relativity in spacetime. In special relativity the transformations that preserve the quadratic form are given an *active interpretation*: they relate the objects or the systems of reference in *relative translational motion*. Analogously also the transformations that preserve the quadratic form (24) or (26) in *C-space* should be given an active interpretation. We have found that among such transformations (rotations in *C-space*) there exist the transformations of the group $SO(4,2)$. Those transformations also should be given an active interpretation as the transformations that relate different physical objects or reference frames. Since in the ordinary relativity we do not impose any constraint on the coordinates of a freely moving object so we should not impose any constraint in *C-space*, or in the subspace V_6 . However, by using the projective coordinate transformation (43), without any constraint such as $\eta^a \eta_a = 0$,

⁶ It is an old observation (Mirman, 2001) that the generators L_{ab} of $SO(4,2)$ can be realized in terms of $\mathbf{1}$, γ_μ , and γ_5 . Lorentz generators are $M_{\mu\nu} = -\frac{i}{4}[\gamma_\mu, \gamma_\nu]$, dilatations are generated by $D = L_{65} = -\frac{1}{2}\gamma_5$, translations by $P_\mu = L_{5\mu} + L_{6\mu} = \frac{1}{2}\gamma_\mu(1 - i\gamma_5)$ and the special conformal transformations by $L_{5\mu} - L_{6\mu} = \frac{1}{2}\gamma_\mu(1 + i\gamma_5)$. This essentially means that the generators are $L_{ab} = -\frac{i}{4}[e_a, e_b]$ with $e_a = (\gamma_\mu, \gamma_5, \mathbf{1})$, where care must be taken to replace commutators $[\mathbf{1}, \gamma_5]$ and $[\mathbf{1}, \gamma_\mu]$ with $2\gamma_5$ and $2\gamma_\mu$.

we arrived at the relation (55) for the line elements. If in the coordinates η^a the line element is constant, then in the coordinates x^a the line element is changing by a scale factor κ which, in general, depends on the evolution parameter τ . The line element does not necessarily relate the events along a particle's worldline. We may consider the line element between two infinitesimally separated events within an extended object where both have the same coordinate label Λ so that $d\Lambda = 0$. Then the 6-dimensional line element $dx^\mu dx^\nu g_{\mu\nu} - \alpha d\Lambda$ becomes the 4-dimensional line element $dx^\mu dx^\nu g_{\mu\nu}$ and, because of (55) it changes with τ when κ does change. This means that the object changes its size, it is moving dilatationally (Pavšič, 1977, 1980). We have thus arrived at a very far reaching observation that the relativity in C -space implies *scale changes of physical objects as a result of free motion, without presence of any forces or such fields as assumed in Weyl theory*. This was advocated long time ago (Pavšič, 1977, 1980), but without recourse to C -space.

An immediate project would be to construct the gauge theory associated with the Clifford algebra of spacetime. This is currently under investigation and has been called the C -space generalization of Maxwell's Electromagnetism which describes the dynamics and couplings of extended objects to antisymmetric tensor fields of arbitrary rank (Aurilia *et al.*, in preparation).

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